Assignment 3

1. Conjuction

The &E and &I rules are simple in form. The &E rule says that from a sentence of the form X&Y, we can infer X and we can infer Y. In this case, & must be the main connective in X&Y. The &I rule says that if both X and Y have been reached in a proof, then at a later line, we may infer X&Y. In both rules &E and &I, the conclusion depends on the unproved assumptions on which the premises of the rule depend. &E and &I are exactly like $\to$E and MT in this regard.

The points to remember in using &E and &I are:

1) In order to use a conjunction, infer each of its components by &E.
2) In order to prove a conjunction, attempt to derive each of its components, and then use &I.

EXAMPLE 1. $P \to Q,\ R \to S \vdash (P \& R) \to (Q \& S)$

Step 1. Since our goal, $(P \& R) \to (Q \& S)$ is a conditional, I will assume P&R and try to prove Q&S.

1	(1)	$P \to Q$	A
2	(2)	$R \to S$	A
3	(3)	$P \& R$	A
1,2,3	(n-1)	$Q \& S$	new goal
1,2	(n)	$(P \& R) \to (Q \& S)$	$\to$I

Step 2. Since what is to be proved is a conjunction, I will try to prove it by deducing each component.

1	(1)	$P \to Q$	A
2	(2)	$R \to S$	A
3	(3)	$P \& R$	A
?		Q	new goal
?		S	new goal
1,2,3	(n-1)	$Q \& S$	&I
1,2	(n)	$(P \& R) \to (Q \& S)$	$\to$I

Step 3. Line 3 is a conjunction; to use it, I infer each of its components P and R. But then I can use P with line 1 and R with line 2.

1	(1)	$P \to Q$	A
2	(2)	$R \to S$	A
3	(3)	$P \& R$	A
3	(4)	P	3 &E
3	(5)	R	3 &E
1,3	(6)	Q	1,4 $\to$E
2,3	(7)	S	2,5 $\to$E
1,2,3	(8)	$Q \& S$	6,7 &I
1,2	(9)	$(P \& R) \to (Q \& S)$	8 $\to$I (3)

EXAMPLE 2 (P&T)→Q, S&T ├ (P→Q)&S

Explanation	Assumptions	Line	Justification
Step 1. Since our goal is a conjunction, I will attempt to prove each conjunct separately and then use &I to combine them.	1	(1) (P&T)→Q	A
	2	(2) S&T	A
		? P→Q	new goal
		? S	new goal
		(P→Q)&S	&I

Explanation	Assumptions	Line	Justification
Step 2. S follows directly from line 2. So that goal is easy to obtain. Our other goal is a conditional, so I will assume its antecedent and try to prove its consequent.	1	(1) (P&T)→Q	A
	2	(2) S&T	A
	2	(3) S	2 &E
	4	(4) P	A
		(n-2) Q	new goal
		(n-1) P→Q	→I
		(n) (P→Q)&S	&I

Explanation	Assumptions	Line	Justification
Step 3. Our new goal is Q which occurs only in line 1. In order to use line 1 to get Q, I would first need to prove P&T and then use →E. In order to get P&T, I first need P and T each by themselves on separate lines and then use &I. I have P already, and T follows from line 2.	1	(1) (P&T)→Q	A
	2	(2) S&T	A
	2	(3) S	2 &E
	4	(4) P	A
	2	(5) T	2&E
	2,4	(6) P&T	4,5 &I
	1,2,4	(7) Q	1,6 →E
	1,2	(8) P→Q	7 →I (4)
	1,2	(9) (P→Q)&S	3,8 &I

2. Disjunction

The vI rule permits the introduction of any sentence as a disjunct, if we have already reached the other disjunct. This is justifiable because a disjunction is true if one of its components is true. Thus all of the following are correct uses of vI:

P	P	P	P	P	~P
PvQ	QvP	Pv~P	~QvP	PvP	Qv~P
~P	P&R	P&R	PvR	P→R	
~Qv~P	(P&R)v~Q	~Qv(P&R)	(PvR)vQ	(P→R)v(R→Q)	

Like →E, MT, &E, and &I, the conclusion of a use of vI depends on the same unproved assumptions as the premise from which it is derived. In a correct use of vI, the main connective of the conclusion will be a disjunction. Thus the following are NOT correct uses of vI:

1	(1) P→R	A		1	(1) ~P	A
1	(2) (PvQ)→R	1 vI		1	(2) ~(~PvQ)	1 vI

Although the rule vI permits us to add anything we want as a disjunct, the way the rule vI is to be used in a particular proof is frequently determined by what we are trying to prove. To see how this works, consider these examples:

EXAMPLE 3. (RvP)→S ├ P→(SvT)

Step 1. Since we are trying to prove a conditional, I will assume its antecedent and try to prove its consequent.	1	(1) (RvP)→S	A
	2	(2) P	A
	1,2	(n-1) SvT	new goal
	1	(n) P→(SvT)	→I (2)

Step 2. We wish to prove SvT which would follow from either S or from T by vI. T does not occur in the premises and is impossible to prove. So I will try to prove S.	1	(1) (RvP)→S	A
	2	(2) P	A
	1,2	(n-2) S	new goal
	1,2	(n-1) SvT	vI
	1	(5) P→(SvT)	→I

Step 3. In order to get S, I will have to use line 1 together with →E. Thus I need to set RvP as my new goal. This is easily gotten from line 2 with vI.	1	(1) (RvP)→S	A
	2	(2) P	A
	2	(3) RvP	2 vI
	1,2	(4) S	1,3→E
	1,2	(5) SvT	4 vI
	1	(6) P→(SvT)	5→I (2)

The <u>vE</u> rule says that from a disjunction and the denial of one of the disjuncts, we can infer the other disjunct. For example, from PvQ and ~P we can infer Q.

EXAMPLE 4. AvB, Bv~C ├ ~B → (A&~C)

Step 1. Since we are trying to prove a conditional, I will assume its antecedent and try to prove its consequent.	1	(1) AvB	A
	2	(2) Bv~C	A
	3	(3) ~B	A
		(n-1) A&~C	new goal
		(n) ~B → (A&~C)	→I

Step 2. Since our goal is a conjunction, I will try to prove each of its conjuncts separately and then put them together with &I.

1	(1) AvB	A
2	(2) Bv~C	A
3	(3) ~B	A
	? A	new goal
	? ~C	new goal
	(n-1) A&~C	&I
	(n) ~B → (A&~C)	→I

Step 3. Each of my two goals is easily obtainable using the vE rule. A follows from lines 1 and 3 while ~C follows from 2 and 3.

1	(1) AvB	A
2	(2) Bv~C	A
3	(3) ~B	A
1,3	(4) A	1,3 vE
2,3	(5) ~C	2,3 vE
1,2,3	(6) A&~C	4,5 &I
1,2	(7) ~B→(A&~C)	6→I (3)
